Journal of Pure and Applied Mathematics: Advances and Applications
Volume 10, Number 2, 2013, Pages 183-189

SEMISIMPLICITY OF THE SECOND DUAL OF BANACH ALGEBRAS

A. ZIVARI-KAZEMPOUR

Department of Mathematics
Ayatollah Borujerdi University
Borujerd
Iran
e-mail: zivari@abru.ac.ir

Abstract

We show that under certain conditions, semisimplicity of the Banach algebra $\mathcal{A}$ implies that of $\mathcal{A}''$, where $\mathcal{A}''$ is the second dual of $\mathcal{A}$ endowed with either Arens product.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra. It is well-known, on the second dual space $\mathcal{A}''$ of $\mathcal{A}$, there are two multiplications, called the first and second Arens products, which make $\mathcal{A}''$ into a Banach algebra (see [1], [4]). By definition, the first Arens product $\Box$ on $\mathcal{A}''$ is induced by the left $\mathcal{A}$-module structure on $\mathcal{A}$. That is, for each $\Phi, \Psi \in \mathcal{A}''$, $f \in \mathcal{A}'$, and $a, b \in \mathcal{A}$, we have

$$\langle \Phi \Box \Psi, f \rangle = \langle \Phi, \Psi \cdot f \rangle, \quad \langle \Psi \cdot f, a \rangle = \langle \Psi, f \cdot a \rangle, \quad \langle f \cdot a, b \rangle = \langle f, ab \rangle.$$

2010 Mathematics Subject Classification: 46H20, 46H25.

Keywords and phrases: Arens regular, semisimple, second dual, multiplicative linear functional.

Received August 19, 2013

Similarly, the second Arens product $\Diamond$ on $\mathcal{A}''$ is defined by considering $\mathcal{A}$ as a right $\mathcal{A}$-module. The Banach algebra $\mathcal{A}$ is said to be *Arens regular*, if $\Phi \square \Psi = \Phi \Diamond \Psi$ on the whole of $\mathcal{A}''$.

The Jacobson radical $J(\mathcal{A})$ of $\mathcal{A}$ is the intersection of the primitive ideals of $\mathcal{A}$. Note that $J(\mathcal{A})$ is the intersection of the maximal modular left ideals of $\mathcal{A}$ by Theorem 1.5.2 of [4], and $\mathcal{A}$ is called *semisimple* if $J(\mathcal{A}) = \{0\}$.

The strong radical $R(\mathcal{A})$ of $\mathcal{A}$ is the intersection of the maximal modular ideals of $\mathcal{A}$, and $\mathcal{A}$ is called *strongly semisimple* if $R(\mathcal{A}) = \{0\}$. For example, let $\mathcal{A} = \mathcal{L}(E)$, where E is an infinite dimensional linear space. Then $\mathcal{A}$ is not strongly semisimple, but $J(\mathcal{A}) = \{0\}$.

A nonzero linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ is said to be *multiplicative* if

$$\varphi(ab) = \varphi(a)\varphi(b) \quad (a, b \in \mathcal{A}).$$

We denote by $\mathfrak{M}(\mathcal{A})$, the set of all multiplicative linear functional on $\mathcal{A}$. If $\mathcal{A}$ is commutative, then by Theorem 5, Section 16 of [2], the maximal modular ideals of $\mathcal{A}$ are the kernels of the multiplicative linear functionals. Therefore, in this case, $J(\mathcal{A}) = R(\mathcal{A})$, and both of them coincide with

$$rad(\mathcal{A}) = \bigcap\{\ker \varphi : \varphi \in \mathfrak{M}(\mathcal{A})\}.$$

Note that $\mathfrak{M}(\mathcal{A})$, is a locally compact Hausdorff space, and it is compact if $\mathcal{A}$ is unital [2].

A bounded net $(e_\alpha)_{\alpha \in I}$ in $\mathcal{A}$ is a bounded approximate identity (BAI for short) if, for each $a \in \mathcal{A}, ae_\alpha \to a$ and $e_\alpha a \to a$. An element $\Phi_0 \in \mathcal{A}''$ is called *mixed unit*, if it is a right unit for $(\mathcal{A}'', \square)$ and a left unit for $(\mathcal{A}'', \Diamond)$. It is well-known that an element $\Phi_0 \in \mathcal{A}''$ is a mixed unit if and only if it is a weak* cluster point of some BAI $(e_\alpha)_{\alpha \in I}$ in $\mathcal{A}$ [2].

Let $\mathcal{A}$ be a Banach algebra with BAI. We say that $\mathcal{A}'$ factors on the left if $\mathcal{A}' = \mathcal{A}' \cdot \mathcal{A}$, and factors on the right if $\mathcal{A}' = \mathcal{A} \cdot \mathcal{A}'$ [8].

Throughout the paper, we identify an element of a Banach space X with its canonical image in X''. Also for closed linear subspace E of X, we write $E^{\perp} = \{f \in X' : f|_E = 0\}$.

Recall that all Banach algebras are assumed to be over the complex field $\mathbb{C}$.

The proof of the following result contained in [11], see also [7].

Theorem 1.1. *Let G be a locally compact group. If G is not discrete, then neither $L^1(G)$ nor $M(G)$ is Arens regular.*

Lemma 1.2. *Let $\mathcal{A}$ be a commutative Banach algebra. Then $\mathcal{A}$ is Arens regular if and only if $\mathcal{A}''$ is commutative.*

Proof. This is straightforward.

2. Semisimplicity of $\mathcal{A}''$

It is well-known that every C^*-algebra $\mathcal{A}$ is Arens regular and semisimple. Now, since the second dual of each C^*-algebra is also a C^*-algebra [4], so it is Arens regular and semisimple. But, in general case, this is fails for Banach algebras. This fact that when Arens regularity of $\mathcal{A}$ implies that of $\mathcal{A}''$ have been studied in [9]. In this note, we investigate the semisimplicity of $\mathcal{A}''$, and prove that under special hypotheses semisimplicity passes from $\mathcal{A}$ to its second dual.

Theorem 2.1. *Let $\mathcal{A}$ be a Banach algebra with BAI, which need not to be commutative. If $J(\mathcal{A}'', \Box) = \{0\}$, then $\mathcal{A}'$ factors on the left.*

Proof. Suppose $\mathcal{A}' \neq \mathcal{A}' \cdot \mathcal{A}$, and let $f \in \mathcal{A}'$, which is not in $\mathcal{A}' \cdot \mathcal{A}$. Since $\mathcal{A}' \cdot \mathcal{A}$ is a closed subspace of $\mathcal{A}'$, by Hahn-Banach theorem, there exist a nonzero element $\Phi \in \mathcal{A}''$ such that $\langle \Phi, f \rangle = 1$ and $\langle \Phi, \mathcal{A}' \cdot \mathcal{A} \rangle = 0.$ Therefore $\Phi \in (\mathcal{A}' \cdot \mathcal{A})^{\perp}$ and so $(\mathcal{A}' \cdot \mathcal{A})^{\perp} \neq \{0\}$, which is contradicts of semisimplicity of $\mathcal{A}''$. Thus, $\mathcal{A}'$ factors on the left.

Note that if $J(\mathcal{A}'', \Box) = \{0\}$, then $\mathcal{A}'$ may not factors on the right. For example, let $\mathcal{A} = \mathcal{K}(c_0)$, the operator algebra of all compact linear operators on the sequence space c_0. Then $J(\mathcal{A}'', \Box) = \{0\}$, by Example 6.2 of [5], but $\mathcal{A}'$ does not factors on the right, by Example 2.5 of [8]. In fact, $\mathcal{A}'$ factors on the right, if $J(\mathcal{A}'', \Diamond) = \{0\}$.

As a consequence of Theorem 2.1, we have the next result which was first proved by Civin and Yood in [3].

Corollary 2.2. *Suppose* $\mathcal{A} = L^1(G)$, *for locally compact abelian group G. If G is not discrete, then* $\mathcal{A}''$ *is not commutative and is not semisimple.*

Corollary 2.3. *Let G be a locally compact abelian group and* $\mathcal{A} = M(G)$. *If G is not discrete, then* $\mathcal{A}''$ *is not commutative and is not semisimple.*

Proof. It follows from Lemma 1.2 that $\mathcal{A}''$ is not commutative, and since $L^1(G)''$ is an ideal in $\mathcal{A}''$, so $\mathcal{A}''$ is not semisimple.

Since the second dual of every Arens regular Banach algebra with BAI, is unital [6], so the following result deduce of Proposition 2.3.6 of [4].

Proposition 2.4. *Let* $\mathcal{A}$ *be a commutative Arens regular Banach algebra with BAI. Then* $\mathfrak{M}(\mathcal{A}'')$ *is compact and non-empty.*

In general, the second dual of semisimple Banach algebra $\mathcal{A}$ may not be semisimple. For example, let $\mathcal{A} = l^1(\mathbb{N})$, with pointwise product. Then $\mathcal{A}$ is commutative and semisimple Banach algebra. By Example 4.1 of [5], $\mathcal{A}''$ is commutative, but it is not semisimple. Note that $\mathcal{A}$ is Arens regular by Lemma 1.2, also it is not reflexive but satisfies $\mathcal{A}''\Box\mathcal{A}'' \subseteq \mathcal{A}$, which shows that $\mathcal{A}$ is an ideal in its second dual.

The next result, which is the main one in the paper, provides a criterion for semisimplicity of the second dual.

Theorem 2.5. *Let $\mathcal{A}$ be a commutative semisimple Banach algebra with a BAI. If $\mathcal{A}$ is Arens regular and is an ideal in the second dual, then $\mathcal{A}''$ is commutative and semisimple.*

Proof. Let Φ and Ψ be arbitrary elements of $\mathcal{A}''$ such that $\Phi \neq \Psi$. So, there exist $g \in \mathcal{A}'$ such that $\Phi(g) \neq \Psi(g)$. By hypotheses $\mathcal{A}'$ factors, therefore $g = f \cdot a$ for some $f \in \mathcal{A}'$ and $a \in \mathcal{A}$. Thus, we have $\Phi(f \cdot a) \neq \Psi(f \cdot a)$, and so $a \cdot \Phi \neq a \cdot \Psi$. Since $\mathcal{A}$ is semisimple, $\mathfrak{M}(\mathcal{A})$ separate the point of $\mathcal{A}$ by Corollary 7, Section 17 of [2], so there exist $\varphi \in \mathfrak{M}(\mathcal{A})$ such that

$$\varphi(a \cdot \Phi) \neq \varphi(a \cdot \Psi).$$

Clearly φ'', the second adjoin of φ is multiplicative linear functional on $\mathcal{A}''$, and since $\mathcal{A}$ is an ideal in the second dual, we have

$$\varphi''(a \cdot \Phi) \neq \varphi''(a \cdot \Psi).$$

It follows that $\varphi''(\Phi) \neq \varphi''(\Psi)$, because ker φ'' is an ideal of $\mathcal{A}''$. Thus, $\mathfrak{M}(\mathcal{A}'')$ separate the point of $\mathcal{A}''$, that is $\mathcal{A}''$ is semisimple.

The below example shows that the hypothesis that $\mathcal{A}$ is Arens regular in Theorem 2.5 is essential.

Example 2.6. Suppose G is compact and abelian group, which is not discrete, and let $\mathcal{A} = L^1(G)$ be its group algebra. Then $\mathcal{A}$ is a commutative and semisimple Banach algebra with BAI. Since G is compact, $\mathcal{A}$ is an ideal in the second dual, as is well-known [10]. By Theorem 1.1, $\mathcal{A}$ is not Arens regular. However, $\mathcal{A}''$ is not commutative and is not semisimple.

We recalled that the spectral radius of element $a \in \mathcal{A}$ is denoted by $r(a)$, and is defined by $r(a) = \lim_n \|a^n\|^{\frac{1}{n}}$.

Theorem 2.7. *Suppose* $\mathcal{A}$ *is a commutative and semisimple Banach algebra, which is complete in its spectral radius norm. Then* $\mathcal{A}''$ *is commutative and semisimple.*

Proof. See [3].

Corollary 2.8. *Let* $\mathcal{A}$ *be a commutative semisimple Banach algebra, which is complete in its spectral radius norm. Then* $\mathcal{A}$ *is Arens regular.*

In general, semisimplicity does not inherits by onto homomorphism. For example, let $\mathcal{A}$ be the Banach algebra $C^m([0, 1])$ of all m times continuously differentiable complex-valued functions on $[0, 1]$ with the norm

$$\|f\| = \sum_{k=0}^{m} \frac{1}{k!} \sup|f^k(x)|.$$

Then $\mathcal{A}$ is commutative and semisimple. The set

$$I = \{f \in \mathcal{A} : f(0) = f'(0) = 0\},$$

is a closed ideal of $\mathcal{A}$. If we define $\alpha(x) = x$, then we see that $\alpha^2 \in I$ and so $(\alpha + I)^2 = \alpha^2 + I = 0$. Therefore,

$$r(\alpha + I) = \lim_n \|(\alpha + I)^n\|^{\frac{1}{n}} = 0.$$

Thus, $\alpha + I \in Rad(\frac{\mathcal{A}}{I})$, but $\alpha + I \neq 0$, which implies $(\frac{\mathcal{A}}{I})$ is not semisimple.

The proof of the following result is straightforward and we omit it.

Theorem 2.9. *Let* $\varphi : \mathcal{A} \to \mathcal{B}$ *be an onto homomorphism between commutative Banach algebras. Suppose for each* $f_1 \in \mathfrak{M}(\mathcal{A})$, *there exists* $f_2 \in \mathfrak{M}(\mathcal{B})$ *such that*

$$f_2 \circ \varphi = f_1.$$

Then semisimplicity of $\mathcal{A}$ *implies that of* $\mathcal{B}$.

References

[1] R. Arens, The adjoint of a bilinear operation, Proc. Amer. Math. Soc. 2 (1951), 839-848.

[2] F. F. Bonsall and J. Duncan, Complete Normed Algebra, Springer-Verlag, New York, 1973.

[3] P. Civin and B. Yood, The second conjugate space of a Banach algebra as an algebra, Pacific J. Math. 11 (1961), 847-870.

[4] H. G. Dales, Banach Algebras and Automatic Continuity, London Mathematical Society Monographs, 2000.

[5] H. G. Dales and A. T. M. Lau, The second duals of Beurling algebras, Mem. Amer. Math. Soc. 177(836) (2005).

[6] F. Ghahramani, R. J. Loy and G. A. Willis, Amenability and weak amenability of second conjugate Banach algebras, Proc. Amer. Math. Soc. 124 (1996), 1489-1497.

[7] C. C. Graham, A B.A.I. proof of the non-Arens regularity of $L^1(G)$, Proc. Amer. Math. Soc. 133 (2004), 163-165.

[8] A. T. M. Lau and A. Ülger, Topological centre of certain dual algebras, Trans. Amer. Math. Soc. 348 (1996), 1191-1212.

[9] A. Sahleh and A. Zivari-Kazempour, Arens regularity of certain class of Banach algebras, Abstract and Applied Analysis (2011), Article ID 680952, 6 pages.

[10] A. Ülger, Arens regularity sometimes implies RNP, Pacific J. Math. 143 (1990), 377-399.

[11] N. J. Young, The irregularity of multiplication in group algebras, Quart. J. Math. 24 (1973), 59-62.

■